

Differential Equations of Order One

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Differential Equations

(1) Definition. A differential equation is an equation which involves differential coefficients or differentials.

Thus,

$$(i) \quad e^x dx + e^y dy = 0 \qquad (ii) \quad \frac{d^2 x}{dt^2} + n^2 x = 0$$

$$(iii) \quad y = x \frac{dy}{dx} + \frac{x}{dy/dx} \qquad (iv) \quad \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2 y}{dx^2}} = c$$

$$(v) \quad \begin{aligned} \frac{dx}{dt} - wy &= a \cos pt, \\ \frac{dy}{dt} + wx &= a \sin pt \end{aligned}$$

$$(vi) \quad x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2u$$

$$(vii) \quad \frac{\partial^2 y}{\partial x^2} = c^2 \frac{\partial^2 y}{\partial t^2}$$

are all examples of differential equations.

(2) Ordinary and Partial Differential Equations

An **ordinary differential equation** is one in which all the differential coefficients have reference to a single independent variable. Thus, equations (i) to (v) are all ordinary differential equations.

A **partial differential equation** is one in which there are two or more independent variables and partial differential coefficients with respect to any of them. Thus, equations (vi) and (vii) are partial differential equations.

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(3) Order and Degree of a Differential Equation

The **order** of a differential equation is the order of the highest derivative appearing in it.

The **degree** of a differential equation is the degree of the highest derivative occurring in it, after the equation has been expressed in a form free from radicals and fractions as far as the derivatives are concerned.

Thus, from the examples above:

- (i) Equation (i) is of the first order and first degree.
- (ii) Equation (ii) is of the second order and first degree.
- (iii) Equation (iii), written as

$$y \frac{dy}{dx} = x \left(\frac{dy}{dx} \right)^2 + x,$$

is clearly of the first order but of the second degree.

- (iv) Equation (iv), written as

$$\left[1 + \left(\frac{dy}{dx} \right)^2 \right]^3 = c^2 \left(\frac{d^2y}{dx^2} \right)^2,$$

is of the second order and second degree.

Formation of a Differential Equation

An ordinary differential equation is formed in an attempt to eliminate certain arbitrary constants from a relation in the variables and constants. It will, however, be seen later that partial differential equations may be formed by the elimination of either arbitrary constants or arbitrary functions. In applied mathematics, every geometrical or physical problem, when translated into mathematical language, often gives rise to a differential equation.

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Example 1 Form the differential equation of the family of curves given by

$$x = A \cos(nt + \alpha).$$

Solution. To eliminate the constants A and α , differentiating twice, we have

$$\frac{dx}{dt} = -nA \sin(nt + \alpha)$$

and

$$\frac{d^2x}{dt^2} = -n^2 A \cos(nt + \alpha) = -n^2 x.$$

Thus,

$$\boxed{\frac{d^2x}{dt^2} + n^2 x = 0}$$

is the desired differential equation, which states that the acceleration varies as the distance from the origin.

Example 2 Obtain the differential equation of all circles of radius a and centre (h, k) .

Solution. Such a circle is

$$(x - h)^2 + (y - k)^2 = a^2. \quad (i)$$

where h and k , the coordinates of the centre, and a are constants.
Differentiating, we have

$$x - h + (y - k) \frac{dy}{dx} = 0$$

and differentiating again,

$$1 + (y - k) \frac{d^2y}{dx^2} + \left(\frac{dy}{dx} \right)^2 = 0.$$

Then

$$y - k = - \frac{1 + \left(\frac{dy}{dx} \right)^2}{\frac{d^2y}{dx^2}}.$$

Also,

$$x - h = -(y - k) \frac{dy}{dx},$$

so

$$x - h = \frac{\frac{dy}{dx} \left[1 + \left(\frac{dy}{dx} \right)^2 \right]}{\frac{d^2y}{dx^2}}.$$

Substituting these in (i) and simplifying, we get

$$\left[1 + \left(\frac{dy}{dx} \right)^2 \right]^3 = a^2 \left(\frac{d^2y}{dx^2} \right)^2. \quad (\text{ii})$$

This is the required differential equation.

Writing (ii) in the form

$$\frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|} = a$$

it states that the radius of curvature of a circle at any point is constant.

Example 3 Obtain the differential equation of the coaxial circles of the system $x^2 + y^2 + 2ax + c^2 = 0$ where c is a constant and a is a variable.

Solution. We have

$$x^2 + y^2 + 2ax + c^2 = 0. \quad (i)$$

Differentiating w.r.t. x ,

$$2x + 2y \frac{dy}{dx} + 2a = 0$$

or

$$2a = -2 \left(x + y \frac{dy}{dx} \right).$$

Substituting in (i),

$$x^2 + y^2 - 2 \left(x + y \frac{dy}{dx} \right) x + c^2 = 0$$

or

$$2xy \frac{dy}{dx} = y^2 - x^2 + c^2$$

which is the required differential equation.

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SOLUTION OF A DIFFERENTIAL EQUATION

A *solution* (or *integral*) of a differential equation is a relation between the variables which satisfies the given differential equation. For example,

$$x = A \cos(nt + \alpha) \quad (1)$$

is a solution of

$$\frac{d^2x}{dt^2} + n^2x = 0 \quad [\text{Example 11.1}] \quad (2)$$

The **general** (or **complete**) **solution** of a differential equation is that in which the number of arbitrary constants is equal to the order of the differential equation. Thus (1) is a general solution (2) as the number of arbitrary constants (A, α) is the same as the order of (2).

A **particular solution** is that which can be obtained from the general solution by giving particular values to the arbitrary constants.

For example,

$$x = A \cos(nt + \pi/4)$$

is the particular solution of the equation (2) as it can be derived from the general solution (1) by putting $\alpha = \pi/4$.

A differential equation may sometimes have an additional solution which cannot be obtained from the general solution by assigning a particular value to the arbitrary constant. Such a solution is called a **singular solution** and is not of much engineering interest.

Linearly independent solution. Two solutions $y_1(x)$ and $y_2(x)$ of the differential equation

$$\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0 \quad (3)$$

are said to be *linearly independent* if $c_1y_1 + c_2y_2 = 0$ such that $c_1 = 0$ and $c_2 = 0$.

If c_1 and c_2 are not both zero, then the two solutions y_1 and y_2 are said to be *linearly dependent*.

If $y_1(x)$ and $y_2(x)$ any two solutions of (3), then their linear combination $c_1y_1 + c_2y_2$ where c_1 and c_2 are constants, is also a solution of (3).

Example 4 Find the differential equation whose set of independent solutions is $[e^x, xe^x]$.

Solution. Let the general solution of the required differential equation be

$$y = c_1 e^x + c_2 x e^x \quad (i)$$

Differentiating (i) w.r.t. x , we get

$$y_1 = c_1 e^x + c_2 (e^x + x e^x)$$

$$\therefore y - y_1 = -c_2 e^x \quad (ii)$$

Again differentiating (ii) w.r.t. x , we obtain

$$y_1 - y_2 = -c_2 e^x \quad (iii)$$

Subtracting (iii) from (ii), we get

$$y - y_1 - (y_1 - y_2) = 0 \quad \text{or} \quad y - 2y_1 + y_2 = 0$$

Geometrical meaning of a differential equation. Consider any differential equation of the first order and first degree

$$\frac{dy}{dx} = f(x, y) \quad (1)$$

If $P(x, y)$ be any point, then (1) can be regarded as an equation giving the value of $dy/dx (= m)$ when the values of x and y are known. Let the value of m at the point $A_0(x_0, y_0)$ derived from (1) be m_0 . Take a neighbouring point $A_1(x_1, y_1)$ such that the slope of A_0A_1 is m_0 . Let the corresponding value of m at A_1 be m_1 .

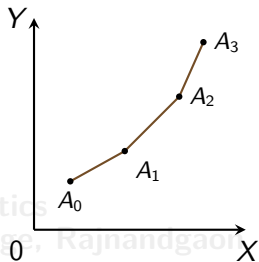


Fig. 11.1

Similarly take a neighbouring point $A_2(x_2, y_2)$ such that the slope of A_1A_2 is m_1 and so on.

If the successive points $A_0, A_1, A_2, A_3 \dots$ are chosen very near one another, the broken curve $A_0A_1A_2A_3 \dots$ approximates to a smooth curve $C[y = \phi(x)]$ which is a solution of (1) associated with the initial point $A_0(x_0, y_0)$. Clearly the slope of the tangent to C at any point and the coordinates of that point satisfy (1).

A different choice of the initial point will, in general, give a different curve with the same property. *The equation of each such curve is thus a **particular solution** of the differential equation (1).* The equation of the whole family of such curves is the *general solution* of (1). The slope of the tangent to any point of each member of this family and the co-ordinates of that point satisfy (1).

Such a simple geometric interpretation of the solutions of a second (or higher) order differential equation is not available.